

GEOMETRIC CONDITIONS FOR INTERPOLATION IN WEIGHTED SPACES OF ENTIRE FUNCTIONS

MYRIAM OUNAIES

ABSTRACT. We use L^2 estimates for the $\bar{\partial}$ equation to find geometric conditions on discrete interpolating varieties for weighted spaces $A_p(\mathbb{C})$ of entire functions such that $|f(z)| \leq Ae^{Bp(z)}$ for some $A, B > 0$. In particular, we give a characterization when $p(z) = e^{|z|}$ and more generally when $\ln p(e^r)$ is convex and $\ln p(r)$ is concave.

INTRODUCTION

Let p be a weight (see Definition 1.1 below) and $A_p(\mathbb{C})$ be the vector space of entire functions satisfying $\sup_{z \in \mathbb{C}} |f(z)|e^{-Bp(z)} < \infty$ for some $B > 0$.

For instance, if $p(z) = |z|$ then $A_p(\mathbb{C})$ is the space of all entire functions of exponential growth. More generally when $p(z) = |z|^\alpha$, $\alpha > 0$ $A_p(\mathbb{C})$ is the space of entire functions of order $< \alpha$ or of order α and of finite type. When $p(z) = \log(1 + |z|^2) + |\operatorname{Im} z|$, $A_p(\mathbb{C})$ is the space of Fourier transforms of distributions with compact support in the real line.

We are concerned with the interpolation problem for $A_p(\mathbb{C})$. That is, finding conditions on a given discrete sequence of complex numbers $V = \{z_j\}_j$ so that, for any sequence of complex numbers $\{w_j\}_j$ with convenient growth conditions, there exists $f \in A_p(\mathbb{C})$ such that $f(z_j) = w_j$, for all j . We will then say that V is an interpolating variety for the weight p . We actually consider the problem with prescribed multiplicities on each z_j , but for the sake of simplicity, we will assume the multiplicities to be equal to 1 in the introduction.

There exists an analytic characterization of interpolating varieties for all weights p satisfying Definition 1.1 (see [3]).

We are interested in finding a geometric description which would enable us to decide whether a discrete sequence is interpolating for $A_p(\mathbb{C})$ by looking at the density of the points. This was done for the weight $p(z) = \log(1 + |z|^2) + |\operatorname{Im} z|$ in [9]. In the present work we will mainly treat radial weights.

The geometric conditions will be given in terms of $N(z, r)$, the integrated counting function of the points of V in the disk of center z and radius r (see Definition 1.6 below).

When p is radial ($p(z) = p(|z|)$) and doubling ($p(2z) \leq 2p(z)$), Berenstein and Li [2] gave a geometric characterization of interpolating varieties for p , namely,

- (i) $N(z_j, |z_j|) = O(p(z_j))$ when $j \rightarrow \infty$;

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(ii) $N(0, r) = O(p(r))$ when $r \rightarrow \infty$.

Hartmann and Massaneda ([6, Theorem 4.3]) gave a proof of this theorem based on L^2 estimates for the solution to the $\bar{\partial}$ -equation of the sufficiency provided that $p(z) = O(|z|^2 \Delta p(z))$. Note that we can always regularize p into a smooth function, see Remark 1.7 below.

In this paper we will give a proof in the same spirit as [6] without the assumption on the Laplacian of p (see Theorem 1.9).

When the above condition on the Laplacian is satisfied, we will prove that (i) is necessary and sufficient (see Theorem 1.10).

In [2], Berenstein and Li also studied rapidly growing radial weights, allowing infinite order functions in $A_p(\mathbb{C})$, as $p(z) = e^{|z|}$, and more generally weights such that $\ln p(e^r)$ is convex. They gave sufficient conditions as well as necessary ones.

We will give a characterization of interpolating varieties for the weight $p(z) = e^{|z|}$ and more generally for weights p such that $p(z) = O(\Delta p(z))$ (see Theorem 1.12) and also for radial p when $\ln p(e^r)$ is convex and $\ln p(r)$ is concave for large r (see Theorem 1.13).

In particular, we will show that V is interpolating for $A_p(\mathbb{C})$, $p(z) = e^{|z|}$, if and only if

$$N(z_j, e) = O(e^{|z_j|}), \text{ when } j \rightarrow \infty.$$

The difficult part in each case is the sufficiency. As in [4, 6, 9], we will follow a Bombieri-Hörmander approach based on L^2 -estimates on the solution to the $\bar{\partial}$ -equation. The scheme will be the following: the condition on the density gives a smooth interpolating function F with a good growth such that the support of $\bar{\partial}F$ is far from the points $\{z_j\}$ (see Lemma 2.6). Then we are led to solve the $\bar{\partial}$ -equation: $\bar{\partial}u = -\bar{\partial}F$ with L^2 -estimates, using a theorem of Hörmander [7]. To do so, we need to construct a subharmonic function U with a convenient growth and with prescribed singularities on the points z_j . Following Bombieri [5], the fact that e^{-U} is not summable near the points $\{z_j\}$ forces u to vanish on the points z_j and we are done by defining the interpolating entire function by $u + F$.

The delicate point of the proof is the construction of the function U . It is done in two steps: first we construct a function U_0 behaving like $\ln |z - z_j|^2$ near z_j with a good growth and with a control on ΔU_0 (the Laplacian of U_0), thanks to the conditions on the density and the hypothesis on the weight itself. Then we add a function W such that ΔW is large enough so that $U = U_0 + W$ is subharmonic.

A final remark about notation:

A , B and C will denote positive constants and their actual value may change from one occurrence to the next.

$F(t) \lesssim G(t)$ means that there exists constants $A, B > 0$, not depending on t such that $F(t) \leq AG(t) + B$ while $F \simeq G$ means that $F \lesssim G \lesssim F$.

The notation $D(z, r)$ will be used for the Euclidean disk of center z and radius r . We will write $\partial f = \frac{\partial f}{\partial z}$, $\bar{\partial} f = \frac{\partial f}{\partial \bar{z}}$. Then $\Delta f = 4\partial\bar{\partial}f$ denotes the Laplacian of f .

To conclude the introduction, the author wishes to thank X. Massaneda for useful talks and remarks.

1. PRELIMINARIES AND MAIN RESULTS

Definition 1.1. A subharmonic function $p : \mathbb{C} \longrightarrow \mathbb{R}_+$, is called a weight if

- (a) $\ln(1 + |z|^2) = O(p(z))$;
- (b) there exist constants $C_1, C_2 > 0$ such that $|z - w| \leq 1$ implies $p(w) \leq C_1 p(z) + C_2$.

Note that condition (b) implies that $p(z) = O(\exp(A|z|))$ for some $A > 0$.

We will say that the weight is “radial” when $p(z) = p(|z|)$ and that it is “doubling” when $p(2r) \lesssim p(r)$.

Let $A(\mathbb{C})$ be the set of all entire functions, we consider the space

$$A_p(\mathbb{C}) = \left\{ f \in A(\mathbb{C}) : \forall z \in \mathbb{C}, |f(z)| \leq A e^{Bp(z)} \text{ for some } A > 0, B > 0 \right\}.$$

Remark 1.2.

- (i) Condition (a) implies that $A_p(\mathbb{C})$ contains all polynomials.
- (ii) Condition (b) implies that $A_p(\mathbb{C})$ is stable under differentiation.

Examples 1.3.

- $p(z) = \ln(1 + |z|^2) + |\operatorname{Im} z|$. Then $A_p(\mathbb{C})$ is the space of Fourier transforms of distributions with compact support in the real line.
- $p(z) = \ln(1 + |z|^2)$. Then $A_p(\mathbb{C})$ is the space of all the polynomials.
- $p(z) = |z|$. Then $A_p(\mathbb{C})$ is the space of entire functions of exponential type.
- $p(z) = |z|^\alpha$, $\alpha > 0$. Then $A_p(\mathbb{C})$ is the space of all entire functions of order $< \alpha$ or of order α and finite type.
- $p(z) = e^{|z|^\alpha}$, $0 < \alpha \leq 1$.

Throughout the paper $V = \{(z_j, m_j)\}_{j \in \mathbb{N}}$ will denote a multiplicity variety, that is, a sequence of points $\{z_j\}_{j \in \mathbb{N}} \subset \mathbb{C}$ such that $|z_j| \rightarrow \infty$, and a sequence of positive integers $\{m_j\}_{j \in \mathbb{N}}$ corresponding to the multiplicities of the points z_k and p will denote a weight.

Definition 1.4. We say that V is an interpolating variety for $A_p(\mathbb{C})$ if for every doubly indexed sequence $\{w_{j,l}\}_{j,0 \leq l < m_j}$ of complex numbers such that, for some positive constants A and B and for all $j \in \mathbb{N}$,

$$\sum_{l=0}^{m_j-1} |w_{j,l}| \leq A e^{Bp(z_j)},$$

we can find an entire function $f \in A_p(\mathbb{C})$, with

$$\frac{f^l(z_j)}{l!} = w_{j,l}$$

for all $j \in \mathbb{N}$ and $0 \leq l < m_j$.

Remark 1.5. (See [1, Proposition 2.2.2]) Thanks to condition (b), we have, for some constants $A > 0$ and $B > 0$,

$$\forall z \in \mathbb{C}, \quad \sum_k \left| \frac{f^{(k)}(z)}{k!} \right| \leq A e^{Bp(z)}.$$

If we consider the space

$$A_p(V) = \left\{ W = \{w_{j,l}\}_{j,0 \leq l < m_j} \subset \mathbb{C} : \forall j, \sum_{l=0}^{m_j-1} |w_{j,l}| \leq A e^{Bp(z_j)} \text{ for some } A > 0, B > 0 \right\}$$

and we define the restriction map by

$$\begin{aligned} \mathcal{R}_V : A_p(\mathbb{C}) &\longrightarrow A_p(V) \\ f &\mapsto \left\{ \frac{f^l(z_j)}{l!} \right\}_{j,0 \leq l \leq m_j-1}, \end{aligned}$$

we may equivalently define the interpolating varieties by the property that \mathcal{R}_V maps $A_p(\mathbb{C})$ onto $A_p(V)$.

Note that $A_p(\mathbb{C})$ can be seen as the union of the Banach spaces

$$A_{p,B}(\mathbb{C}) = \{f \in A(\mathbb{C}), \|f\|_B = \sup_{z \in \mathbb{C}} |f(z)| e^{-Bp(z)} < \infty\}$$

and has a structure of an (LF)-space with the inductive limit topology. The same can be said about $A_p(V)$.

The problem we are considering is to find conditions on V so that it is an interpolating variety for $A_p(\mathbb{C})$.

In order to state the geometric conditions, we define the counting function and the integrated counting function:

Definition 1.6. For $z \in \mathbb{C}$ and $r > 0$ we set

$$n(z, r) = \sum_{|z - z_j| \leq r} m_j$$

and

$$\begin{aligned} N(z, r) &= \int_0^r \frac{n(z, t) - n(z, 0)}{t} dt + n(z, 0) \ln r \\ &= \sum_{0 < |z - z_j| \leq r} m_j \ln \frac{r}{|z - z_j|} + n(z, 0) \ln r. \end{aligned}$$

Remark 1.7. The weight p may be regularized as in [6, Remark 2.3] by replacing p by its average over the disc $D(z, 1)$. Thus we may suppose p to be of class \mathcal{C}^2 when needed.

Before stating our results we recall that if V is interpolating for $A_p(\mathbb{C})$, then the following two conditions are necessary:

$$(1) \quad \exists A > 0, \exists B > 0, \forall j \in \mathbb{N}, \quad N(z_j, e) \leq A p(z_j) + B$$

and when p is radial,

$$(2) \quad \exists A > 0, \exists B > 0, \forall r > 0, \quad N(0, r) \leq A p(r) + B.$$

See Theorem 2.1 and Proposition 2.3 for the proof.

In condition (i), we may replace $N(z_j, e)$ by $N(z_j, c)$ with any constant $c > 1$. We are now ready to state our main results. We begin by giving sufficient conditions for a discrete variety V to be interpolating for radial weights.

Theorem 1.8. *Assume the weight p to be radial. If condition (1) holds and*

$$(3) \quad \exists A > 0, \exists B > 0 \quad \forall r > 0, \quad \int_0^r n(0, t) dt \leq A p(r) + B$$

then V is interpolating for $A_p(\mathbb{C})$.

We note that

$$\int_0^r n(0, t) dt = \sum_{|z_j| \leq r} m_j(r - |z_j|) \leq r N(0, r).$$

Consequently, by condition (2), we see that $\int_0^r n(0, t) dt \leq A r p(r) + B$, for some $A > 0$ and $B > 0$ is a necessary condition.

Adapting our method to the doubling case we find the characterization given by Berenstein and Li [2, Corollary 4.8]:

Theorem 1.9. *Assume p to be radial and doubling.*

V is interpolating for $A_p(\mathbb{C})$ if and only if conditions (2) holds and

$$(4) \quad \exists A > 0, \exists B > 0 \quad \forall j \in \mathbb{N}, \quad N(z_j, |z_j|) \leq A p(z_j) + B.$$

The theorem holds if we replace $N(z_j, |z_j|)$ by $N(z_j, C|z_j|)$ for any constant $C > 0$. Note that radial and doubling weights satisfy $p(r) = O(r^\alpha)$ for some $\alpha > 0$. In other words, they have at most a polynomial growth. Examples of radial and doubling weights are $p(z) = |z|^\alpha (\ln(1 + |z|^2))^\beta$, $\alpha > 0, \beta \geq 0$, but for $p(z) = |z|^\alpha$, we can give a better result:

Theorem 1.10. *Assume that $p(z) = O(|z|^2 \Delta p(z))$ and*

(b') $\exists C_1 > 0, \exists C_2 > 0$, such that $|z - w| \leq |z|$ implies $p(w) \leq C_1 p(z) + C_2$.

V is interpolating for $A_p(\mathbb{C})$ if and only if condition (4) holds.

Remark 1.11. It is easy to see that radial and doubling weights satisfy condition (b').

Theorem 1.10 applies to $p(z) = |z|^\alpha$, $\alpha > 0$. For this weight and with the extra assumption that there is a function $f \in A_p(\mathbb{C})$ vanishing on every z_j with multiplicity m_j , it was shown in ([10, Theorem 3]) that condition (4) is sufficient and necessary.

Next we are interested in the case where p grows rapidly, allowing infinite order functions in $A_p(\mathbb{C})$. A fundamental example is $p(z) = e^{|z|}$.

In [2], Berenstein and Li studied this weight and more generally those for which $\ln p(e^r)$ is convex. They gave sufficient conditions (Corollaries 5.6 and Corollary 5.7) as well as necessary ones (Theorem 5.14, Corollary 5.15).

The following result gives a characterization in particular for the weight $p(z) = e^{|z|}$ and more generally for $p(z) = e^{|z|^\alpha}$, $\alpha \geq 1$.

Theorem 1.12. Assume that $p(z) = O(\Delta p(z))$.

V is interpolating for $A_p(\mathbb{C})$ if and only if condition (1) holds.

The next theorem will give a characterization when p is radial, $q(r) = \ln p(e^r)$ is convex and $\frac{r}{q'(\ln r)} = \frac{p(r)}{p'(r)}$ is increasing (for large r). If we set $u(r) = \ln p(r)$, we have $\frac{p(r)}{p'(r)} = \frac{1}{u'(r)}$. Thus, the last condition means that $u(r)$ is concave for large r . We recall that the convexity of q implies that $p(r) \geq Ar + B$, for some $A, B > 0$ (see [2, Lemma 5.8]).

The weights $p(z) = |z|^\alpha$, $\alpha > 0$ and $p(z) = e^{|z|}$ satisfy these conditions. Examples of weights also satisfying these conditions but not those of Theorems 1.9, 1.10 or 1.12 are $p(z) = e^{|z|^\alpha}$, $0 < \alpha < 1$ and $p(z) = e^{[\log(1+|z|^2)]^\beta}$, $\beta > 1$.

Theorem 1.13. Assume that p is a radial weight and that for a certain $r_0 > 0$ it satisfies the following properties

- $q(r) := \ln p(e^r)$ is convex on $[\ln r_0, \infty[$;
- $q'(\ln r_0) > 0$ and $\frac{r}{q'(\ln r)}$ is increasing on $[r_0, \infty[$.

Then V is interpolating for $A_p(\mathbb{C})$ if and only if the following condition holds:

$$(5) \quad \exists A > 0, \exists B > 0, \quad \forall |z_j| \geq r_0, N(z_j, \max(\frac{|z_j|}{q'(\ln |z_j|)}, e)) \leq Ap(z_j) + B.$$

The theorem holds if we replace $\frac{|z_j|}{q'(\ln |z_j|)}$ by $\frac{C|z_j|}{q'(\ln |z_j|)}$ for any constant $C > 0$.

When $p(z) = |z|^\alpha$, conditions (5) and (4) are the same and when $p(z) = e^{|z|}$, conditions (5) and (1) are the same.

As immediate corollaries of Theorem 1.13, we have the following

Corollary 1.14. Let $p(z) = e^{|z|^\alpha}$, $0 < \alpha < 1$. V is interpolating for $A_p(\mathbb{C})$ if and only if the following condition holds:

$$(6) \quad \exists A > 0, \exists B > 0, \quad \forall j, \quad N(z_j, |z_j|^{1-\alpha}) \leq Ap(z_j) + B.$$

Corollary 1.15. Let $p(z) = e^{[\ln(1+|z|^2)]^\beta}$, $\beta \geq 1$. V is interpolating for $A_p(\mathbb{C})$ if and only if the following condition holds:

$$(7) \quad \exists A > 0, \exists B > 0, \quad \forall j, \quad N(z_j, |z_j|[\ln(1+|z_j|^2)]^{1-\beta}) \leq Ap(z_j) + B.$$

2. GENERAL RESULTS ABOUT THE INTERPOLATION THEORY

For the sake of completeness, we include in this section some standard results about interpolating varieties. Let us begin by well known necessary conditions.

Theorem 2.1. [10, Theorem 1] Assume that V is an interpolating variety for $A_p(\mathbb{C})$. Let R_j be positive numbers satisfying

$$(8) \quad |z - z_j| \leq R_j \implies p(z) \leq C_1 p(z_j) + C_2$$

where C_1 and C_2 are positive constants not depending on j . Then the following condition holds:

$$(9) \quad \exists A > 0, \exists B > 0, \quad \forall j, \quad N(z_j, R_j) \leq A p(z_j) + B.$$

Remark 2.2. In view of property (b) of the weight p , $R_j = 1$ satisfies condition (8). In the case where p satisfies the stronger condition (b') (as in Theorem 1.10) we are allowed the larger numbers $R_j = |z_j|$.

Proposition 2.3. *Assume that V is an interpolating variety for $A_p(\mathbb{C})$. Then the following condition is satisfied*

$$(10) \quad \exists A > 0, \exists B > 0, \forall R > 0, \quad N(0, R) \leq A \max_{|z|=R} p(z) + B.$$

The proof of this proposition is implicitly contained in [2, page 26].

Definition 2.4. We say that V is weakly separated if there exist constants $A > 0$ and $B > 0$ such that

$$(11) \quad \frac{1}{\delta_j^{m_j}} \leq A e^{Bp(z_j)}$$

for all $j \in \mathbb{N}$, where

$$\delta_j := \inf \left\{ 1, \frac{1}{2} \inf_{k \neq j} |z_j - z_k| \right\}$$

is called the separation radius.

Lemma 2.5. *If (1) holds then V is weakly separated.*

Proof. Fix $j \in \mathbb{N}$ and let $z_l \neq z_j$ be such that $|z_j - z_l| = \inf_{k \neq j} |z_j - z_k|$. If $|z_j - z_l| \geq 2$, then $\delta_j = 1$. Otherwise, $2\delta_j = |z_j - z_l|$ and the following inequalities hold

$$N(z_l, e) \geq \sum_{0 < |z_k - z_l| \leq e} m_k \ln \frac{e}{|z_k - z_l|} \geq m_j \ln \frac{e}{|z_j - z_l|} = m_j \ln \frac{e}{2\delta_j} \geq \ln \frac{1}{\delta_j^{m_j}}.$$

Since by condition (1) and property (b) of the weight,

$$N(z_l, e) \leq A + Bp(z_l) \leq A + Bp(z_j),$$

we readily deduce the desired estimate. ■

We will follow the same scheme as in [6, 9], first constructing a smooth interpolating function with the right growth.

Lemma 2.6. *Suppose V is weakly separated. Given $W = \{w_{j,l}\}_{j \in \mathbb{N}, 0 \leq l \leq m_j-1} \in A_p(V)$, there exists a smooth function F such that*

- for some constants $A > 0$ and $B > 0$, $|F(z)| \leq A e^{Bp(z)}$, $|\bar{\partial}F(z)| \leq A e^{Bp(z)}$ for all $z \in \mathbb{C}$;
- the support of $\bar{\partial}F$ is contained in the union of the annuli

$$A_j = \{z \in \mathbb{C} : \frac{\delta_j}{2} \leq |z - z_j| \leq \delta_j\};$$

- $\frac{F^{(l)}(z_j)}{l!} = w_{j,l}$ for all $j \in \mathbb{N}$, $0 \leq l \leq m_j - 1$.

A suitable function F is of the form

$$F(z) = \sum_j \mathcal{X}\left(\frac{|z - z_j|^2}{\delta_j^2}\right) \sum_{l=0}^{m_j-1} w_{j,l}(z - z_j)^l,$$

where \mathcal{X} is a smooth cut-off function with $\mathcal{X}(x) = 1$ if $|x| \leq 1/4$ and $\mathcal{X}(x) = 0$ if $|x| \geq 1$. See [6] or [9] for details of the proof.

Now, when looking for a holomorphic interpolating function of the form $f = F + u$, we are led to the $\bar{\partial}$ -problem

$$\bar{\partial}u = -\bar{\partial}F,$$

which we solve using Hörmander's theorem [8, Theorem 4.2.1].

The interpolation problem is then reduced to the following:

Lemma 2.7. *If V is weakly separated and if there exists a subharmonic function U satisfying for certain constants $A, B > 0$,*

- (i) $U(z) \leq Ap(z) + B$ for all $z \in \mathbb{C}$;
- (ii) $-U(z) \leq Ap(z) + B$ for z in the support of $\bar{\partial}F$;
- (iii) $U(z) \simeq m_j \ln |z - z_j|^2$ near z_j ,

then V is interpolating.

Proof. The weak separation gives a smooth interpolating function F (see Lemma 2.6). From Hörmander's theorem [7, Theorem 4.4.2], we can find a C^∞ function u such that $\bar{\partial}u = -\bar{\partial}F$ and, denoting by $d\lambda$ the Lebesgue measure,

$$(12) \quad \int_{\mathbb{C}} \frac{|u(w)|^2 e^{-U(w) - Ap(w)}}{(1 + |w|^2)^2} d\lambda(w) \leq \int_{\mathbb{C}} |\bar{\partial}F|^2 e^{-U(w) - Ap(w)} d\lambda(w).$$

By property (a) of the weight p , there exists $C > 0$ such that

$$\int_{\mathbb{C}} e^{-Cp(w)} d\lambda(w) < \infty.$$

Thus, using (ii) of the lemma, and the estimate on $|\bar{\partial}F(z)|^2$, we see that the last integral in (12) is convergent if A is large enough. By condition (iii), near z_j , $e^{-U(w)}(w - z_j)^l$ is not summable for $0 \leq l \leq m_j - 1$, so we have necessarily $u^{(l)}(z_j) = 0$ for all j and $0 \leq l \leq m_j - 1$ and

consequently, $\frac{f^{(l)}(z_j)}{l!} = w_{j,l}$.

Now, we have to verify that f has the desired growth. By the mean value inequality,

$$|f(z)| \lesssim \int_{D(z,1)} |f(w)| d\lambda(w) \lesssim \int_{D(z,1)} |F(w)| d\lambda(w) + \int_{D(z,1)} |u(w)| d\lambda(w).$$

Let us estimate the two integrals that we denote by I_1 and I_2 . For $w \in D(z, 1)$,

$$|F(w)| \lesssim e^{Bp(w)} \lesssim e^{Cp(z)}.$$

Hence

$$I_1 \lesssim e^{Cp(z)}.$$

To estimate I_2 we use Cauchy-Schwarz inequality

$$I_2^2 \leq J_1 J_2$$

where

$$J_1 = \int_{D(z,1)} |u(w)|^2 e^{-U(w)-Bp(w)} d\lambda(w), \quad J_2 = \int_{D(z,1)} e^{U(w)+Bp(w)} d\lambda(w).$$

By property (a) of p we have

$$J_1 \lesssim \int_{\mathbb{C}} |u(w)|^2 e^{-U(w)-Bp(w)} d\lambda(w) \lesssim \int_{\mathbb{C}} \frac{|u(w)|^2 e^{-U(w)-Bp(w)}}{(1+|w|^2)^2} d\lambda(w) < +\infty$$

if $B > 0$ is chosen big enough.

To estimate J_2 we use condition (i) of the lemma and property (b) of the weight p . For $w \in D(z, 1)$,

$$e^{U(w)+Bp(w)} \leq e^{Cp(w)} \lesssim e^{Ap(z)}.$$

We easily deduce that $J_2 \lesssim e^{Ap(z)}$ and finally that $f \in A_p(\mathbb{C})$. ■

3. PROOFS OF THE MAIN THEOREMS

We will use a smooth cut-off function \mathcal{X} with $\mathcal{X}(x) = 1$ if $|x| \leq 1/4$ and $\mathcal{X}(x) = 0$ if $|x| \geq 1$.

Remark 3.1. In the proofs of the sufficiency part, we may need to assume that for all j , we have $|z_j| \geq a$ for a suitable $a > 0$. This will be done without loss of generality up to a linear transform and in view of property (b) of the weight.

Proof of Theorem 1.8.

By Lemma 2.5, condition (1) implies the weak separation. So we are done if we construct a function U satisfying the conditions of Lemma 2.7.

Set $\mathcal{X}_j(z) = \mathcal{X}(|z - z_j|^2)$.

In order to construct the desired function we begin by defining

$$U_0(z) = \sum_j m_j \mathcal{X}_j(z) \ln |z - z_j|^2.$$

Note that there is locally a finite number of non vanishing terms in the sum and that each term (and consequently U_0) is nonpositive. It is also clear that $U_0(z) - m_j \ln |z - z_j|^2$ is continuous near z_j .

We want to estimate $-U_0$ on the support of $\bar{\partial}F$, and the “lack of subharmonicity” of U_0 , then we will add a correcting term to obtain the function U of the lemma.

Suppose z is in the support of $\bar{\partial}F$. We want to show that $-U_0(z) \lesssim p(z)$. Let k be the unique integer such that $\frac{\delta_k}{2} \leq |z - z_k| \leq \delta_k$. Then

$$-U_0(z) \leq 2 \sum_{|z-z_j| \leq 1} m_j \ln \frac{1}{|z - z_j|} = 2m_k \ln \frac{1}{|z - z_k|} + 2 \sum_{j \neq k, |z-z_j| \leq 1} m_j \ln \frac{1}{|z - z_j|}.$$

Using that $|z - z_k| \geq \frac{\delta_k}{2}$ and that for $j \neq k$ we have

$$|z_k - z_j| \leq |z - z_j| + |z - z_k| \leq 2|z - z_j|,$$

we obtain that

$$(13) \quad -U_0(z) \leq 2 \ln \frac{1}{\delta_k^{m_k}} + 2N(z_k, 2) \lesssim p(z_k) \lesssim p(z).$$

The last inequalities follows from condition (1), the weak separation (11) and property (b) of the weight p .

Now we want to get a lower bound on $\Delta U_0(z)$. We have

$$\begin{aligned} \Delta U_0(z) &= \sum_j m_j \mathcal{X}_j(z) \Delta \ln |z - z_j|^2 \\ &\quad + 8 \operatorname{Re} \left(\sum_j m_j \bar{\partial} \mathcal{X}_j(z) \partial \ln |z - z_j|^2 \right) + 4 \sum_j m_j \partial \bar{\partial} \mathcal{X}_j(z) \ln |z - z_j|^2. \end{aligned}$$

The first sum is a positive measure and on the supports of $\bar{\partial} \mathcal{X}_j$ and $\partial \bar{\partial} \mathcal{X}_j$, we see that $1/2 \leq |z - z_j| \leq 1$. Consequently, for a certain constant $\gamma > 0$ we have

$$\Delta U_0(z) \geq -\gamma(n(z, 1) - n(z, 1/2)) \geq -\gamma n(z, 1) \geq -\gamma(n(0, |z| + 1) - n(0, |z| - 1)).$$

We set $n(0, t) = 0$ if $t < 0$,

$$f(t) = \int_{t-1}^{t+1} n(0, s) ds, \quad g(t) = \int_0^t f(s) ds \text{ and } W(z) = g(|z|).$$

Let us compute the Laplacian of W , taking the derivatives in the sense of distributions

$$\Delta W(z) = \frac{1}{|z|} g'(|z|) + g''(|z|) \geq g''(|z|) = f'(|z|) = n(0, |z| + 1) - n(0, |z| - 1).$$

The function U defined by

$$U(z) = U_0(z) + \gamma W(z)$$

is then clearly subharmonic. On the other hand, using condition (3) and (b) we have the following inequalities:

$$f(s) \leq 2n(0, s + 1), \quad W(z) = g(|z|) \leq 2 \int_1^{|z|+1} n(0, s) ds \lesssim p(z).$$

Since $U_0 \leq 0$ it is clear that U satisfies condition (i) of Lemma 2.7. Using the estimate (13) and the fact that W is nonnegative we see that U satisfies condition (ii). Finally as condition (iii) is also already fulfilled by U_0 and W is continous, it is also fulfilled by U . \blacksquare

Proof of Theorem 1.9.

Necessity. In view of Remark 1.11 and Remark 2.2, we apply Theorem 2.1 with $R_j = |z_j|$ and we readily obtain the necessity of (4).

Condition (2) is necessary by Proposition 2.3.

Sufficiency. By Lemma 2.5, condition (4) implies the weak separation. We will proceed as in Theorem 1.8, constructing a function U satisfying (i), (ii) and (iii) from Lemma 2.7. Thanks to

the doubling condition, we can control the weight p in discs $D(z_j, |z_j|)$ instead of just $D(z_j, e)$ in the general case. We will construct U_0 as in the previous theorem, except that we now take \mathcal{X}_j 's with supports of radius $\simeq |z_j|$:

Set

$$\mathcal{X}_j(z) = \mathcal{X}\left(\frac{16|z - z_j|^2}{|z_j|^2}\right),$$

and introduce the negative function

$$U_0(z) = \sum_j m_j \mathcal{X}_j(z) \ln \frac{16|z - z_j|^2}{|z_j|^2}.$$

When z is in the support of $\bar{\partial}F$, let k be the unique integer such that $\frac{\delta_k}{2} \leq |z - z_k| \leq \delta_k$. Repeating the estimate on $-U_0(z)$, we have

$$-U_0(z) \leq 2 \sum_{|z - z_j| \leq \frac{|z_j|}{4}} m_j \ln \frac{|z_j|}{4|z - z_j|} \leq 2m_k \ln \frac{|z_k|}{\delta_k} + 2 \sum_{0 < |z_k - z_j| \leq \frac{|z_j|}{2}} m_j \ln \frac{|z_j|}{2|z_k - z_j|}.$$

We have $\frac{|z_j|}{2} \leq |z_k|$ whenever $|z_k - z_j| \leq \frac{|z_j|}{2}$. We deduce the inequalities

$$(14) \quad -U_0(z) \leq 2 \ln \frac{1}{\delta_k^{m_k}} + 2N(z_k, |z_k|) \lesssim p(z_k) \lesssim p(z).$$

Again, the last inequalities follow from condition (1), the weak separation (11) and property (b) of the weight p .

We estimate $\Delta U_0(z)$ as before except that now $|\bar{\partial}\mathcal{X}_j(z)| \lesssim \frac{1}{|z_j|}$ and $|\partial\bar{\partial}\mathcal{X}_j(z)| \lesssim \frac{1}{|z_j|^2}$. On the support of these derivatives, $\frac{|z_j|}{8} \leq |z - z_j| \leq \frac{|z_j|}{4}$ and $\frac{|z|}{2} \leq |z_j| \leq 2|z|$. We deduce that

$$\Delta U_0(z) \gtrsim -\frac{n(0, 2|z|) - n(0, \frac{|z|}{2})}{|z|^2}.$$

To construct the correcting term W set

$$f(t) = \int_0^t n(0, s)ds, \quad g(t) = \int_0^t \frac{f(s)}{s^2}ds \quad \text{and} \quad W(z) = g(2|z|).$$

Finally, to estimate the Laplacian of W , we write $t = 2|z|$. We have

$$\Delta W(z) = 4 \left(\frac{1}{t} g'(t) + g''(t) \right) = \frac{4}{t^2} \left(f'(t) - \frac{f(t)}{t} \right)$$

and

$$f(t) = \int_0^t n(0, s)ds = \int_0^{\frac{t}{4}} n(0, s)ds + \int_{\frac{t}{4}}^t n(0, s)ds \leq \frac{t}{4} n\left(0, \frac{t}{4}\right) + t \left(1 - \frac{1}{4}\right) n(0, t).$$

Thus,

$$f'(t) - \frac{f(t)}{t} = n(0, t) - \frac{f(t)}{t} \geq \frac{1}{4} \left(n(0, t) - n\left(0, \frac{t}{4}\right) \right)$$

and

$$\Delta W(z) \gtrsim \frac{n(0, 2|z|) - n(0, \frac{|z|}{2})}{|z|^2}.$$

Now, the desired function will be of the form

$$U(z) = U_0(z) + \gamma W(z),$$

where γ is a positive constant sufficiently large. The following inequalities are easy to see:

$$f(t) \leq tn(0, t), \quad g(t) \leq \int_0^t \frac{n(0, s)}{s} ds = N(0, s).$$

Thus, by condition (2) and the doubling condition,

$$0 \leq W(z) \leq N(0, 2|z|) \lesssim p(2z) \lesssim p(z).$$

We conclude that U satisfies all the desired conditions. ■

Proof of Theorem 1.10.

Necessity. Recalling Remark 2.2, we apply once again Theorem 2.1 to deduce the necessity of condition (4).

Sufficiency. The proof is the same as for Theorem 1.9, we only change the estimate on ΔU_0 and the correcting term W . Let us have a new look at $\Delta U_0(z)$.

$$(15) \quad \Delta U_0(z) \gtrsim - \sum_{|z - z_j| \leq \frac{|z_j|}{4}} \frac{m_j}{|z|^2}.$$

If the sum is not empty, let z_k be the point appearing in the sum with the largest modulus. For all z_j such that $|z - z_j| \leq \frac{|z_j|}{4}$, we have

$$|z_j - z_k| \leq |z - z_k| + |z - z_j| \leq \frac{|z_k|}{4} + \frac{|z_j|}{4} \leq \frac{|z_k|}{2}.$$

We deduce that

$$\Delta U_0(z) \gtrsim - \frac{n(z_k, \frac{|z_k|}{2})}{|z|^2}.$$

Besides,

$$n(z_k, \frac{|z_k|}{2}) \leq m_k + \frac{1}{\ln 2} \sum_{0 < |z_j - z_k| \leq \frac{|z_k|}{2}} m_j \ln \frac{|z_k|}{|z_j - z_k|} \lesssim N(z_k, |z_k|) \lesssim p(z_k).$$

Note that $|z - z_k| \leq \frac{|z_k|}{4}$ implies that $|z - z_k| \leq |z|$. Thus by condition (b') we have $p(z_k) \lesssim p(z)$. Finally we get

$$\Delta U_0(z) \gtrsim - \frac{p(z)}{|z|^2} \gtrsim -\Delta p(z).$$

Then we take

$$U(z) = U_0(z) + \gamma p(z)$$

where γ is a positive constant chosen large enough. ■

Proof of Theorem 1.12. We already know by Theorem 2.1 that condition (1) is necessary.

Let us consider the function U_0 that we constructed in the proof of Theorem 1.8. Again, we only change the estimate on ΔU_0 and the correcting term W . We find

$$(16) \quad \Delta U_0(z) \gtrsim -n(z, 1).$$

If $n(z, 1) \neq 0$, let z_k be in $D(z_k, 1)$. Then

$$n(z, 1) \leq n(z_k, 2) \leq m_k + \frac{1}{1 - \ln 2} \sum_{0 < |z_k - z_j| < 2} m_j \ln \frac{e}{|z_k - z_j|} \lesssim N(z_k, e) \lesssim p(z_k) \lesssim p(z).$$

The function

$$U(z) = U_0(z) + \gamma p(z),$$

with $\gamma > 0$ large enough has the desired properties. ■

Proof of Theorem 1.13.

We set $c := \inf(q'(\ln r_0), 1)$ and $\psi(r) = \frac{r}{q'(\ln r)}$.

Claim 3.2.

(i) Let $r \geq 2r_0$. Then $c\psi(r) \leq r$ and

$$|x| \leq c \frac{\psi(r)}{2} \text{ implies that } \frac{\psi(r)}{2} \leq \psi(r + x) \leq 2\psi(r);$$

(ii) For all $r \geq r_0$,

$$p(r + \psi(r)) \leq ep(r).$$

Assuming this claim true for the moment, let us proceed with the proof of Theorem 1.13. *Necessity.* In view of Theorem 2.1, it suffices to show that $R_j = \psi(|z_j|)$ satisfy condition (8). Let $|z_j| \geq r_0$ and w be such that $|w - z_j| \leq R_j$. Thus $|w| \leq |z_j| + \psi(|z_j|)$ and as a consequence of (ii) of the claim, we obtain

$$p(w) \leq p(|z_j| + \psi(|z_j|)) \leq ep(|z_j|).$$

Sufficiency. We may assume that $|z_j| \geq r_0$ for all j (see Remark 3.1). We apply Lemma 2.5 to deduce that V is weakly separated. We repeat the proof of Theorems 1.10 and 1.9, replacing $|z_j|$ by $c\psi(|z_j|)$. More precisely, we set

$$\mathcal{X}_j(z) = \mathcal{X} \left(\frac{16|z - z_j|^2}{c^2\psi(|z_j|)^2} \right)$$

and we define the negative function

$$U_0(z) = \sum_j m_j \mathcal{X}_j(z) \ln \frac{16|z - z_j|^2}{c^2\psi(|z_j|)^2}.$$

We use (i) of Claim 3.2 to obtain that whenever $|z - z_j| \leq \frac{c\psi(|z_j|)}{2}$, we have

$$\begin{aligned}\psi(|z|) &\leq \psi(|z_j| + |z - z_j|) \leq 2\psi(|z_j|); \\ \frac{\psi(|z_j|)}{2} &\leq \psi(|z_j| - |z - z_j|) \leq \psi(|z|).\end{aligned}$$

Adapting the inequality (14) when $\frac{\delta_k}{2} \leq |z - z_k| \leq \delta_k$ and applying condition (5) we find

$$-U_0(z) \leq 2 \ln \frac{1}{\delta_k^{m_k}} + 2N(z_k, c\psi(|z_k|)) \lesssim p(z_k) \lesssim p(z).$$

Let us now find a lower bound for ΔU_0 on \mathbb{C} . By analogy to (15) and the inequalities following we obtain

$$\Delta U_0(z) \gtrsim - \sum_{|z - z_j| \leq \psi(|z_j|)c/4} \frac{1}{\psi(|z_j|)^2} \gtrsim - \frac{N(z_l, c\psi(|z_l|))}{(\psi(|z|))^2}$$

where z_l is one of the points appearing in the sum with the largest modulus. Recall that

$$|z - z_l| \leq \psi(|z_l|)c/4 \leq \psi(|z|)c/2 \leq \psi(|z|)$$

and consequently from (ii) of Claim 3.2 we deduce that

$$p(z_l) \leq p(|z| + |z - z_l|) \leq p(|z| + \psi(|z|)) \leq ep(z).$$

Finally, we apply condition (5)

$$\Delta U_0(z) \gtrsim - \frac{p(z)}{\psi(|z|)^2}.$$

Let us compute the Laplacian of $p(z) = e^{q(\ln |z|)}$ in terms of the convex function q . Setting $r = |z|$,

$$\Delta p(z) = \frac{p'(r)}{r} + p''(r) = \frac{[q'(\ln r)]^2}{r^2} p(r) + \frac{q''(\ln r)}{r^2} p(r) \geq \frac{[q'(\ln r)]^2}{r^2} p(r) = \frac{p(r)}{[\psi(r)]^2}.$$

We readily deduce that

$$\Delta U_0(z) \geq -\gamma \Delta p(z)$$

for some $\gamma > 0$. As in the preceding proofs, the function $U(z) = U_0(z) + \gamma p(z)$ satisfies the properties stated in Lemma 2.7. ■

Proof of Claim 3.2. Let $r > r_0$. A computation gives

$$\psi'(r) = \frac{q'(\ln r) - q''(\ln r)}{[q'(\ln r)]^2}.$$

Recall that q'' is nonnegative. Besides as q' is increasing we have $q'(\ln r) \geq q'(\ln r_0) \geq c$. We deduce that $0 \leq \psi'(r) \leq \frac{1}{c}$. Note that we also have the inequality $c\psi(r) \leq r$.

Assume now $r \geq 2r_0$ and let $|x| \leq \frac{c\psi(r)}{2}$. Then

$$|r + x| \geq r - \frac{c\psi(r)}{2} \geq \frac{r}{2} \geq r_0.$$

We can now use the mean value theorem and the preceding estimates to write

$$|\psi(r+x) - \psi(r)| \leq \frac{|x|}{c} \leq \frac{\psi(r)}{2}.$$

We easily deduce (i).

To prove (ii), put $u(t) = q(\ln t) = \ln p(t)$. We have $u'(t) = \frac{1}{\psi(t)} \leq \frac{1}{\psi(r)}$ for all $t \geq r \geq r_0$ because ψ is increasing on $[r_0, \infty[$. Applying the mean value theorem again, we obtain $u(r + \psi(r)) - u(r) \leq 1$. We deduce that

$$p(r + \psi(r)) = e^{u(r+\psi(r)) - u(r)} p(r) \leq e p(r). \quad \blacksquare$$

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INSTITUT DE RECHERCHE MATHÉMATIQUE AVANCÉE, UNIVERSITÉ LOUIS PASTEUR 7 RUE RENÉ DESCARTES, 67084 STRASBOURG CEDEX, FRANCE.

E-mail address: ounaies@math.u-strasbg.fr